A p-ADIC BERTINI THEOREM FOR UNIPOTENT LOCAL SYSTEMS

CHRIS LAZDA

Abstract. In this short note we prove a version of Bertini's theorem for unipotent rigid fundamental groups, stating that for every smooth, projective, geometrically connected variety X over an infinite perfect field k of characteristic $p > 0$, there exists a smooth, projective, geometrically connected curve $C \subset X$ such that the induced map on rigid fundamental groups is surjective.

Introduction

Let k be a perfect field of characteristic $p > 0$, and let K be a complete, discretely valued field of characteristic 0 with residue field k. Then for any geometrically connected variety X/k, i.e. geometrically connected, separated scheme of finite type over k, we can consider the Tannakian category $\mathrm{Isoc}^\dagger(X/K)$ of overconvergent isocrystals on X/K (see for example [1]). If $x \in X(k)$, we let $\pi_1^{\mathrm{rig}}(X, x)$ denote the rigid fundamental group of X at x, that is the Tannaka dual of the subcategory $\mathcal{N}\mathrm{Isoc}^\dagger(X/K)$ of unipotent isocrystals with respect to the fibre functor x^*. We will also denote by $\pi_1^{\mathrm{Isoc}^\dagger}(X, x)$ the Tannaka dual of $\mathrm{Isoc}^\dagger(X/K)$ with respect to x^*. The goal of this short note is to indicate a proof of the following theorem.

Theorem. *Let $X \subset \mathbb{P}^n_k$ be a smooth, projective, geometrically connected variety over k. Suppose that $H \subset \mathbb{P}^n_k$ is a hyperplane such that $Y = X \cap H$ is non-empty, smooth and geometrically connected. Then for any $y \in Y(k)$ the induced map*

$$\pi_1^{\mathrm{rig}}(Y, y) \to \pi_1^{\mathrm{rig}}(X, y) \tag{1}$$

is surjective.

We mention the following application.

Corollary. *Let X/k be a smooth, projective, geometrically connected variety over k. Then there exists a smooth, projective, geometrically connected curve $C \subset X$ such that for all $c \in C(k)$ the induced map*

$$\pi_1^{\mathrm{rig}}(C, c) \to \pi_1^{\mathrm{rig}}(X, c) \tag{2}$$

is surjective.

Proof. First assume that k is infinite. Then by applying the usual Bertini theorem, we may repeatedly cut X by hyperplanes to get a smooth, projective, geometrically connected curve $C \subset X$. By the above theorem, at each stage the induced map on rigid fundamental groups is surjective.

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If k is finite, then Poonen's results [17] imply that there is a sequence of smooth, geometrically connected, non-empty subvarieties $X = X_0 \supset X_1 \supset \ldots \supset X_n = C$ with C a curve, such that each X_{i+1} is the intersection of X_i with a hyperplane for some projective embedding of X_i. Thus we may proceed exactly as before to show that the map induced by $C \to X$ on rigid fundamental groups is surjective. $\square$

1. Unipotent and relatively unipotent fundamental groups

In this section we prove that a certain sequence of fundamental groups arising from a smooth and proper morphism of k-varieties is exact. Let $f : X \to S$ be a smooth, proper morphism of quasi-projective k-varieties of relative dimension d, with geometrically connected fibres. Assume that S is an open subscheme of $\mathbb{P}^1_k$. Let $s \in S(k)$ be a k-valued point, and let $i_s : X_s \to X$ denote the inclusion of the fibre over s. We will denote by $\mathcal{N}_f\mathrm{Isoc}^\dagger(X/K)$ the full subcategory of $\mathrm{Isoc}^\dagger(X/K)$ consisting of iterated extensions of isocrystals pulled back from S via f. For $x \in X(k)$ we denote by $G_X(x)$ the Tannaka dual of this category with respect to x^*. In this section we wish to prove the following result.

Theorem 1.1. *Let $x \in X_s(k)$ be a k-valued point of X mapping to $s \in S(k)$, and assume that $f : X \to S$ admits a section p. Then the sequence of affine group schemes*

$$\pi_1^{\mathrm{rig}}(X_s, x) \xrightarrow{i_{s*}} G_X(x) \xrightarrow{f_*} \pi_1^{\mathrm{Isoc}^\dagger}(S, s) \longrightarrow 1 \tag{3}$$

is exact.

The map f_* is surjective, since the section p of f induces a section p_* of f_*, and since the composite functor $\mathrm{Isoc}^\dagger(S/K) \to \mathcal{N}\mathrm{Isoc}(X_s/K)$ is trivial, we have $f_* \circ i_{s*} = 1$, the trivial homomorphism.

Lemma 1.2. *For any smooth, proper morphism $f : X \to S$ as above, there is a functor $f_* : \mathcal{N}_f\mathrm{Isoc}^\dagger(X/K) \to \mathrm{Isoc}^\dagger(S/K)$, adjoint to f^*. If S is reduced to a point, then this becomes the canonical adjunction between $V \mapsto V \otimes_K \mathscr{O}^\dagger_{X/K}$ and $E \mapsto H^0_{\mathrm{rig}}(X, E)$. Moreover, for all points $s : \mathrm{Spec}(k) \to S$ there is an isomorphism of functors $s^* \circ f_*(-) \cong H^0_{\mathrm{rig}}(X_s, i_s^*(-))$, where $i_s : X_s \to X$ denotes the inclusion of the fibre over s.*

Proof. Let $\mathcal{V}$ be the ring of integers of K, and choose an embedding $X \hookrightarrow \mathbb{P}^n_k$. Let $\mathscr{P} = \widehat{\mathbb{P}}^n_{\mathcal{V}} \times_{\mathcal{V}} \widehat{\mathbb{P}}^1_{\mathcal{V}}$, and $X \hookrightarrow \mathscr{P}$ the locally closed immersion given by the product of $X \hookrightarrow \mathbb{P}^n_k$ and $f : X \to S \subset \mathbb{P}^1_k$. Let $\bar{X}$ be the closure of X in $\mathscr{P}$, and let $g : \mathscr{P} \to \widehat{\mathbb{P}}^1_{\mathcal{V}}$ denote the second projection, this is smooth and proper. We let $H \subset \mathbb{P}^1_k$ be the complement of S, P the special fibre of $\mathscr{P}$, and $T \subset P$ the inverse image of H. We will use freely the language of arithmetic $\mathscr{D}$-modules, as developed by Berthelot and Caro, see for example the series of papers [2–4] and [5–10, 12].

We will let $\mathrm{Isoc}^{\dagger\dagger}(X, \bar{X}/K) = \mathrm{Isoc}^{\dagger\dagger}(\mathscr{P}, T, \bar{X}/K)$ denote the category of overcoherent isocrystals on $(X, \bar{X}/K)$, see for example Section 3 of [11]. According to *loc. cit.* there is a functor

$$\mathrm{sp}_{X \hookrightarrow \mathscr{P}, T, +} : \mathrm{Isoc}^\dagger(X/K) \to \mathrm{Isoc}^{\dagger\dagger}(X, \bar{X}/K) \tag{4}$$

which is an equivalence of categories. We will denote by $D^b_{\mathrm{isoc}}(X,\bar{X}/K)$ the full subcategory of $D^b_{\mathrm{surcoh}}(X,\bar{X}/K)=D^b_{\mathrm{surcoh}}(\mathscr{P},T,\bar{X}/K)$ consisting of those objects whose cohomology sheaves lie in $\mathrm{Isoc}^{\dagger\dagger}(X,\bar{X}/K)$. We also have the categories $\mathrm{Isoc}^{\dagger\dagger}(S,\mathbb{P}^1_k/K)=\mathrm{Isoc}^{\dagger\dagger}(\widehat{\mathbb{P}}^1_{\mathcal{V}},H/K)$ and $D^b_{\mathrm{isoc}}(S,\mathbb{P}^1_k/K)\subset D^b_{\mathrm{surcoh}}(S,\mathbb{P}^1_k/K)=D^b_{\mathrm{surcoh}}(\widehat{\mathbb{P}}^1_{\mathcal{V}},H/K)$, as well as an equivalence $\mathrm{Isoc}^{\dagger\dagger}(S,\mathbb{P}^1_k/K)\cong\mathrm{Isoc}^{\dagger}(S/K)$.

According to Proposition 4.2.9 of [5], we have functors

$$\begin{aligned} f_+ &: D^b_{\mathrm{surcoh}}(X,\bar{X}/K)\to D^b_{\mathrm{surcoh}}(S,\mathbb{P}^1_k/K) \\ f^! &: D^b_{\mathrm{surcoh}}(S,\mathbb{P}^1_k/K)\to D^b_{\mathrm{surcoh}}(X,\bar{X}/K) \end{aligned} \tag{5}$$

and we can define

$$f^+ : D^b_{\mathrm{surcoh}}(S,\mathbb{P}^1_k/K)\to D^b_{\mathrm{surcoh}}(X,\bar{X}/K) \tag{6}$$

as in Lemme 3.1.17 of [11]. The proof of *loc. cit.* shows that f_+ is right adjoint to f^+. Hence if E,F are overcoherent isocrystals on $(X,\bar{X}/K)$ and $(S,\mathbb{P}^1_k/K)$ respectively, we get a natural identification

$$\mathrm{Hom}_{D^b_{\mathrm{surcoh}}(X,\bar{X}/K)}(f^+F,E)=\mathrm{Hom}_{D^b_{\mathrm{surcoh}}(S,\mathbb{P}^1_k/K)}(F,f_+E). \tag{7}$$

According to Proposition 4.2.4 of *loc. cit.*, we know that f^+F is isomorphic to f^*F concentrated in degree d, where we have used the identifications of categories of overcoherent and overconvergent isocrystals to transport the natural pullback functor via f from the latter category to the former.

Remark 1.3. It may seem that Proposition 4.2.4 of [11] says that f^+F is isomorphic to f^*F concentrated in degree zero. A careful reading of the paper, however, shows that for $\theta=(a,b,f)$ a morphism of triples as considered in the proposition, the functor θ^+ is actually isomorphic to the functor a^+ shifted by the relative dimension of the morphism a.

Also, according to Théorème 4.2.12 of [5], $f_+E\in D^b_{\mathrm{isoc}}(S,\mathbb{P}^1_k/K)$. Hence we get an identification

$$\mathrm{Hom}_{\mathrm{Isoc}^{\dagger\dagger}(X,\bar{X}/K)}(f^*F,E)=\mathrm{Hom}_{D^b_{\mathrm{isoc}}(S,\mathbb{P}^1_k)/K}(F,f_+E[-d]). \tag{8}$$

Now consider the point $s:\mathrm{Spec}(k)\to(S,\mathbb{P}^1_k)$, and let $f_s:X_s\to\mathrm{Spec}(k)$ denote the structure morphism of the fibre over s. Since overcoherent isocrystals on $(S,\mathbb{P}^1_k/K)$ are flat as modules over $\mathcal{O}_{\widehat{\mathbb{P}}^1_{\mathcal{V}}}({}^\dagger H)_{\mathbb{Q}}$ (see Proposition 4.4.2 of [2]), it follows from combining Proposition 3.1.7 of [11] with Théorème 4.4.2 of [5] that we get a functorial isomorphism $\mathcal{H}^i(f_{s+}i_s^*E)\cong s^*\mathcal{H}^i(f_+E)$.

Claim. Let $F\in\mathrm{Isoc}^{\dagger\dagger}(X_s,/K)$. Then there is a functorial isomorphism $\mathcal{H}^i(f_{s+}F)\cong H^{i+d}_{\mathrm{rig}}(X_s,F)$.

Proof. As in the proof of Lemme 7.3.4 of [8], this follows from combining 4.6.3.6 of [4] with cohomological descent for an open affine covering of X_s. □

Hence $f_+E[-d]$ has cohomology concentrated is positive degrees, and so defining $f_*E=\mathcal{H}^{-d}(f_+E)$ we get an identification

$$\mathrm{Hom}_{\mathrm{Isoc}^{\dagger\dagger}(X,\bar{X}/K)}(f^*F,E)=\mathrm{Hom}_{\mathrm{Isoc}^{\dagger\dagger}(S,\mathbb{P}^1_k)/K}(F,f_*E) \tag{9}$$

as well as a 'base change' isomorphism of functors $s^*\circ f_*\cong H^0_{\mathrm{rig}}(X_s,-)\circ i_s^*$.

Now exploiting the equivalence between $\mathrm{Isoc}^\dagger$ and $\mathrm{Isoc}^{\dagger\dagger}$ gives us the required functor $f_* : \mathcal{N}_f\mathrm{Isoc}^\dagger(X/K) \to \mathrm{Isoc}^\dagger(S/K)$. □

Proof of Theorem 1.1. According to Lemma 1.3 and Proposition 1.4 in Chapter 1.1 of [18], we need to prove the following.

(1) Let $E \in \mathcal{N}_f\mathrm{Isoc}^\dagger(X/K)$. Then i_s^*E is constant if and only if $E \cong f^*F$ for some $F \in \mathrm{Isoc}^\dagger(S/K)$.
(2) Let $E \in \mathcal{N}_f\mathrm{Isoc}^\dagger(X/K)$, and let $F_0 \subset i_s^*E$ denote the largest constant subobject. Then there exists $E_0 \subset E$ with $F_0 = i_s^*E_0$.

If $E \in \mathcal{N}_f\mathrm{Isoc}^\dagger(X/K)$ is such that i_s^*E is constant, consider the counit $f^*f_*E \to E$ of the adjunction between f^* and f_*. By base change, this restricts to the counit $H^0_{\mathrm{rig}}(X_s, i_s^*E) \otimes_K \mathcal{O}^\dagger_{X_s/K} \to i_s^*E$ of the adjunction

$$- \otimes_K \mathcal{O}^\dagger_{X_s/K} : \mathrm{Vec}_K \rightleftarrows \mathcal{N}\mathrm{Isoc}^\dagger\,(X_s/K) : H^0_{\mathrm{rig}}(X_s, -)\ . \tag{10}$$

on the fibre, which is an isomorphism since i_s^*E is constant. Hence $f^*f_*E \to E$ must be an isomorphism by rigidity. In general we know that $H^0(X_s, i_s^*E) \otimes_K \mathcal{O}^\dagger_{X_s/K}$ is the largest trivial subobject F_0 of i_s^*E. Hence $E_0 = f^*f_*E$ is a subobject of E restricting to F_0. □

2. Proof of the Main Theorem

In this section we use the results of the previous section to deduce the main theorem. We start with a couple of simple lemmas.

Lemma 2.1. *Suppose that Y is a smooth, proper, geometrically connected k-variety, $j : U \to Y$ the inclusion of an open subscheme and $u \in U(k)$. Then the induced map*

$$\pi_1^{\mathrm{rig}}(U, u) \to \pi_1^{\mathrm{rig}}(Y, u) \tag{11}$$

is surjective.

Proof. By Proposition 2.21 of [15] it suffices to show that $j^* : \mathcal{N}\mathrm{Isoc}^\dagger(Y/K) \to \mathcal{N}\mathrm{Isoc}^\dagger(U/K)$ is fully faithful, with image stable under taking subquotients. But this is just Theorem 5.2.1 and Proposition 5.3.1 of [13]. □

Lemma 2.2. *Let $p \in \mathbb{P}^1_k(k)$. Then $\pi_1^{\mathrm{Isoc}^\dagger}(\mathbb{P}^1_k, p) = \{1\}$.*

Proof. We must show that every convergent isocrystal on $\mathbb{P}^1_k$ is constant. Since the functor from convergent isocrystals on $\mathbb{P}^1_k$ to coherent modules with integrable connection on $\mathbb{P}^{1,\mathrm{an}}_K$ is fully faithful (Theorem 2.15 of [16]), it suffices to show that every coherent module with integrable connection on $\mathbb{P}^{1,\mathrm{an}}_K$ is constant.

We first claim that every coherent module with integrable connection on the rigid analytification of a smooth, projective K-variety X is in fact algebraic. Indeed, the underlying coherent module is algebraic by rigid analytic GAGA, call it V^{an}. The integrable conenction on V^{an} is given by a stratification, i.e. a collection of isomorphisms on the nth infinitesimal neighbourhoods $(X^{\mathrm{an}})^{(n)}$ of X^{an} in the

diagonal. Let $X^{(n)}$ denote the nth infinitesimal neighbourhood of X in the diagonal. Since $(X^{\mathrm{an}})^{(n)} = (X^{(n)})^{\mathrm{an}}$, and $X^{(n)}$ is proper over K, rigid analytic GAGA applied on each $X^{(n)}$ gives us a collection of isomorphisms which amount to an algebraic stratification on V, so the integrable connection is algebraic.

Remark 2.3. It is worth noting that GAGA holds for all projective K-schemes - not necessarily reduced, see GAGA Satz 4.7 and 5.1 of [14].

It thus suffices to show that every algebraic integrable connection on $\mathbb{P}^1_K$ is constant. Any such connection is defined over a finitely generated extension of $\mathbb{Q}$ contained in K, and hence we may assume that we can embed K into the complex numbers $\mathbb{C}$. Since every integrable connection on $\mathbb{P}^1_{\mathbb{C}}$ is constant, it suffices to show that an integrable connection $\mathcal{E}$ on $\mathbb{P}^1_K$ is constant if its base change to $\mathbb{C}$ is. But $\mathcal{E}$ is constant if and only if the dimension of $H^0_{\mathrm{dR}}(\mathbb{P}^1_K, \mathcal{E})$ over K is equal to the rank of $\mathcal{E}$. Using the fact that de Rham cohomology commutes with arbitrary ground field extensions, and the corresponding criterion for constancy of $\mathcal{E}_{\mathbb{C}}$, we see that $\mathcal{E}$ is constant if and only if $\mathcal{E}_{\mathbb{C}}$ is constant. $\square$

Theorem 2.4. *Let $X \subset \mathbb{P}^n_k$ be smooth and projective. Suppose that $H \subset \mathbb{P}^n_k$ is a hyperplane such that $Y = X \cap H$ is non-empty, smooth and geometrically connected. Then for any $y \in Y(k)$ the induced map*

$$\pi_1^{\mathrm{rig}}(Y, y) \to \pi_1^{\mathrm{rig}}(X, y) \tag{12}$$

is surjective.

Proof. Since formation of π_1^{rig} commutes with finite extension of the ground field, and subjectivity of a map of affine group schemes can be checked after passing to a finite extension, we may make such and extension and hence assume (by Bertini's hyperplane section theorem) that there exists a hyperplane $H' \subset \mathbb{P}^n_k$ such that $y \in Y \cap H'$ and $Z = Y \cap H'$ is smooth. Let $\pi : \widetilde{X} \to X$ be the blow-up of X along Z, and $\widetilde{Y} \to Y$ the proper transform of π along $Y \to X$.

Claim. For any $x' \in \pi^{-1}(x)$ the induced map $\pi_* : \pi_1^{\mathrm{rig}}(\widetilde{X}, x') \to \pi_1^{\mathrm{rig}}(X, x)$ is an isomorphism.

Proof. We want to show that the induced functor $\mathcal{N}\mathrm{Isoc}^\dagger(X/K) \to \mathcal{N}\mathrm{Isoc}^\dagger(\widetilde{X}/K)$ is an equivalence of categories, and it suffices to prove the correponding statement for the category of all overconvergent isocrystals (since then the unipotent parts of both categories will coincide). According to Proposition 5.3.6 of [13], for every overconvergent isocrystal E on $\widetilde{X}$, there exists a unique overconvergent isocrystal F on X such that $E \cong \pi^* F$, thus the functor

$$\pi^* : \mathrm{Isoc}^\dagger(X/K) \to \mathrm{Isoc}^\dagger(\widetilde{X}/K) \tag{13}$$

is essentially surjective. It is automatically faithful, hence we must demonstrate that it is full. So let $\pi^* E \to \pi^* F$ be a morphism. Since $\pi : \widetilde{X} \setminus \pi^{-1}(Z) \to X \setminus Z$ is an isomorphism, this induces a morphism $E|_{X\setminus Z} \to F|_{X \setminus Z}$ which by full faithfullness of pullback via an open immersion observed above must come from a morphism $E \to F$. $\square$

The same is true for $\widetilde{Y} \to Y$, and hence it suffices to show that $\pi_1^{\mathrm{rig}}(\widetilde{Y},\tilde{y}) \to \pi_1^{\mathrm{rig}}(\widetilde{X},\tilde{y})$ is surjective. The pencil of hyperplane sections spanned by $Y = X \cap H$ and $X \cap H'$ furnishes a projective map $a : \widetilde{X} \to \mathbb{P}^1_k$ whose generic fibre is smooth, and the preimage of y with respect to π gives a section σ of a. Let $b : V \to U$ be the smooth locus of a, note that $\widetilde{Y} \subset V$ is a fibre of b.

As before, let $\mathcal{N}_b\mathrm{Isoc}^\dagger(V/K)$ denote the full subcategory of $\mathrm{Isoc}^\dagger(V/K)$ consisting of iterated extensions of objects pulled back from $\mathrm{Isoc}^\dagger(U/K)$, and $G_V(\tilde{y})$ its Tannaka dual with respect to the base point $\tilde{y}$. Since b has geometrically connected fibres, the results of Section 1 imply that the induced sequence of group schemes

$$\pi_1^{\mathrm{rig}}(\widetilde{Y},\tilde{y}) \to G_V(\tilde{y}) \to \pi_1^{\mathrm{Isoc}^\dagger}(U,a(\tilde{y})) \to 1 \tag{14}$$

is exact. We also have a diagram of affine group schemes

$$\begin{tikzcd}
\pi_1^{\mathrm{rig}}(\widetilde{Y},\tilde{y}) \arrow[r] \arrow[d, equal] & G_V(\tilde{y}) \arrow[r] \arrow[d] & \pi_1^{\mathrm{Isoc}^\dagger}(U,a(\tilde{y})) \arrow[r] \arrow[d] \arrow[l, bend right, "\sigma_*"'] & 1 \\
\pi_1^{\mathrm{rig}}(\widetilde{Y},\tilde{y}) \arrow[r] & G_{\widetilde{X}}(\tilde{y}) \arrow[r] & \pi_1^{\mathrm{Isoc}^\dagger}(\mathbb{P}^1_k,a(\tilde{y})) \arrow[r] \arrow[l, bend right, "\sigma_*"'] & 1
\end{tikzcd} \tag{15}$$

where the groups appearing in the bottom row are defined in an analogous manner.

Claim. The middle vertical arrow $G_V(\tilde{y}) \to G_{\widetilde{X}}(\tilde{y})$ is surjective.

Proof. There is a commutative diagram of affine group schemes

$$\begin{tikzcd}
\pi_1^{\mathrm{Isoc}^\dagger}(V,\tilde{y}) \arrow[r] \arrow[d] & G_V(\tilde{y}) \arrow[d] \\
\pi_1^{\mathrm{Isoc}^\dagger}(X,\tilde{y}) \arrow[r] & G_X(\tilde{y})
\end{tikzcd} \tag{16}$$

with horizontal maps surjective. Since $V \to X$ is an open immerison, the same argument as in Lemma 2.1 (replacing $\mathcal{N}\mathrm{Isoc}^\dagger$ by $\mathrm{Isoc}^\dagger$ - Kedlaya's Theorem works for both) shows that the left vertical map is surjective, hence so it the right vertical map. □

Now the fact that the sequence in Equation (14) is exact implies that $G_V(\tilde{y})$ is generated by the images of $\pi_1^{\mathrm{rig}}(\widetilde{Y},\tilde{y})$ and $\pi_1^{\mathrm{Isoc}^\dagger}(U,a(\tilde{y}))$. Since $G_V(\tilde{y}) \to G_{\widetilde{X}}(\tilde{y})$ is surjective, it follows that $G_{\widetilde{X}}(\tilde{y})$ is generated by the images of $\pi_1^{\mathrm{rig}}(\widetilde{Y},\tilde{y})$ and $\pi_1^{\mathrm{Isoc}^\dagger}(\mathbb{P}^1_k,a(\tilde{y}))$. Since $\pi_1^{\mathrm{Isoc}^\dagger}(\mathbb{P}^1_k,a(\tilde{y})) = \{1\}$, we conclude that $\pi_1^{\mathrm{rig}}(\widetilde{Y},\tilde{y}) \to G_{\widetilde{X}}(\tilde{y})$ is surjective. The Tannakian category $\mathrm{Isoc}^\dagger(\mathbb{P}^1_k/K)$ is trivial, so $\mathcal{N}_a\mathrm{Isoc}^\dagger(\widetilde{X}/K) = \mathcal{N}\mathrm{Isoc}^\dagger(\widetilde{X}/K)$ and hence $G_{\widetilde{X}}(\tilde{y}) = \pi_1^{\mathrm{rig}}(\widetilde{X},\tilde{y})$. Thus $\pi_1^{\mathrm{rig}}(\widetilde{Y},\tilde{y}) \to \pi_1^{\mathrm{rig}}(\widetilde{X},\tilde{y})$ is surjective as claimed.

□

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Department of Mathematics, 180 Queen's Gate, Imperial College, London, SW7 2AZ, United Kingdom

E-mail address: `c.lazda10@imperial.ac.uk`